I  INTRODUCTION

I.1 First Price-Dominance of Futures over Forwards

In a previous article\(^1\) we have established a price difference between forward and future, which we briefly justify as follows.

In mature markets, the future bid-ask spread fluctuates strictly between the bid and the ask of bank arbitrageurs. This bid and this ask make up the forward quotes that the banks would list should the future not exist. The future buy price is consequently strictly less than the forward’s, and the future sell price strictly greater than the forward’s. The future is therefore more attractive than the forward in terms of price.

The forward bid-ask spread constitutes the no pure arbitrage range of the future.

I.2 Second Price-Dominance of Futures over Forwards

In this paper we set up a second argument to the advantage of the future.

The Cost of Carry Model posits that the price of a term contract, whether future or forward, is worth its underlying plus the cost of its carry minus its revenues. But when the contract is a future it is endowed with a presettlement feature: we will show here that it is proper to adjust the future price with the value of an implicit option said of early unwinding, whose underlying asset is the pure arbitrage hedge, with zero strike price and expiring along with the future.

In the presence of market bid-ask spreads, the option assimilates to either a call on a cash and carry hedge, or a put on the reverse hedge. The no pure arbitrage range,
spanning from the forward ask quote minus the call to the forward bid quote plus the bid, is
thus narrower than the forward bid-ask spread. As a consequence the early unwinding
option will be a second argument in favor of the price-dominance of futures over forwards.

I.3 The Brennan and Schwartz Article

Research has seldom focused on this option. Only articles from Merrick\(^2\) and Brennan &
Schwartz\(^3\) stand out but they have not given rise to any further extension in the academic
world. Two articles likely to deal with the same matter are cited by Brennan and Schwartz:
MacKinlay and Ramaswamy’s in the Review of Financial Studies is not available in France
due to its publication date (1988), and the one of the two authors in Essays in Financial
Economics (1988) does not exist within French libraries databases.

If Merrick indeed measures real opportunities of presettlement he nevertheless does not
express formally any option. On the other hand Brennan and Schwartz have tried to do it
and a short analysis of their work is achieved hereafter.

I.3.1 Summary

Here we summarize and criticize main points of the Brennan and Schwartz article.

The article is about optimal arbitrage strategies settlable before term, with transaction
costs and position limits. The special case of the S&P 500 future is being studied. The price
difference between futures and forwards is considered as negligible. The article begins by
describing the mark-to-market value of an arbitrage position in the absence of transaction
costs, and borrows from two other researchers (Stoll and Whaley) a cost structure which it
subtracts from this value. Then it posits the arbitrage constraints binding the opening and
closing options of a position, with and without limits. Then it assumes that the mark-to-
market value without cost follows a brownian bridge, from this deduces a process
descriptive differential equation, and establishes the early exercise conditions. It points out
the origin of the empirical data, and principally that the mark-to-markets come from the
work of two other researchers (McKinlay and Ramaswamy). It calibrates the brownian
bridge process on the empirical data by estimating a pseudo-likelyhood maximum, then
carries out the numerical resolution of the differential equation while adopting the
parameters stemming from the calibration. Lastly the article estimates the options on

\(^1\) “Une nouvelle comparaison entre future et forward”, Gilles Desvilles, Banque et Marchés, 2000.

\(^2\) “Early unwindings and rollovers of stock index futures arbitrage programs”, John J. Merrick, The Journal of
Futures Markets, volume 9, 1989.

\(^3\) “Arbitrage in stock index futures”, Michael J. Brennan and Eduardo S. Schwartz, Journal of Business, volume
63, 1990.
positions with limits, and gives details of the profits yielded by their ex-post optimal exercice. The conclusion regrets the exogenous feature of the mark-to-market process, whose calibration provides unstable parameters.

I.3.2 Critical Analysis

There are three main critiques of the Brennan and Schwartz work.

- Their intermediary literal calculus shows that they have implicitly supposed that the futures are not fungible. The two authors assimilate an early close-out strategy with two pure arbitrages held to expiration, that we will label static in this paper. They have probably been driven to this by their assumption that a pure arbitrage payoff is cashed at the inception and not at the expiration date. This contrasts with our approach where the futures fungibility will play an essential role in order to differentiate the dynamic arbitrage from the static arbitrage.

- The pricing of the two presettlement options — called early close-out options — brings in an existential option — called arbitrage option — measuring the right to arbitrage or to do nothing, and owned by every trader. The early close-out options cannot be dissociated from the existential option for their price equations are interwoven and can be solved only simultaneously. We reject the validity of the existential option for we consider that it is as free as the air. An option has only value if it is contracted through a purchase, a hedge or any other tangible commitment. Owing to the mentioned interweaving, the pricing of the early close-out options is wrong.

- Lastly, Brennan and Schwartz get a differential equation, describing the options process, that they solve with the finite difference method. Getting this equation makes an improper use of the argument of risk-neutrality. They apply the argument directly to each option and not to a hedge portfolio including the option being considered, as theory requires. Hence the resulting differential equation is found to be erroneous, and accordingly the proposed option values not reliable.

I.3.3 Common Purpose

We share with Brennan and Schwartz a fourth critique of their work, which feeds the ultimate conclusion of their article:

The real challenge, however, remains to endogenize the stochastic behavior of the simple arbitrage opportunity given the nature of transaction costs and the structure of the market.

It is this challenge that the present paper tries to take up. Instead of forcing the mark-to-market of an arbitrage to follow a process unusual in finance and popping up from a handbook of Statistics, we first prefer to get its financial expression (part II) in which the
market variables will then be modelled in a way generally accepted in finance (part III), as for example a stock price is with a log-normal process. A conclusion (part IV) ends this paper.

II A GENERAL AND REALISTIC FRAMEWORK OF THE FUTURES ARBITRAGE

The framework exposed hereafter is generic for it deals with a term contract written on an undefined financial asset. It is realistic because it imbeds the quote spreads of all the financial instruments and interest rates.

II.1 General Framework

II.1.1 The Spot Assets

The spot or cash interest rates, i.e. spanning a period starting from a contemporaneous date, are denoted \( i \).

The lending or investment rates are indexed with \( b \), as bid, and the borrowing’s are indexed with \( a \), as ask. Of course \( i_b < i_a \) for fear of offering an arbitrage opportunity through simultaneously lending and borrowing.

The rates are compounded on an annual basis.

The underlying financial asset is denoted \( U \). \( U \) can be traded in a market, for sale at the quote \( U_b \) — bid — and for purchase at the quote \( U_a \) — ask — . Of course \( U_b < U_a \) for fear of offering an arbitrage opportunity through simultaneously buying and selling \( U \). We designate the quote spread with \([U_b : U_a]\).

For sake of clarity of the writing and without loss of generality, the financial asset is supposed to pay no revenue\(^4\).

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\(^4\) The incidence of the revenue is modelled in “Arbitrage sur les marchés à terme”, Gilles Desvilles, PhD Dissertation, March 1998. The revenue significantly complicates the writing but doesn’t distort the reach of the results presented in this study. These can thus be generalized to the case with revenue.
II.1.2 The Future Contract

There exists a future contract on U, denoted F. This contract is auctioned for purchase at $F_a$ and for sale at $F_b$. The expiration or maturity date of F is denoted $T$.

F is supposed to settle by delivery. The adjustments related to a cash settlement would not modify fundamentally the results to follow.

**Basis**

The *basis* is the difference between the contract and its underlying:

$$B = U - F$$

The underlining of $B$ distinguishes it from a note classically written $B$. The basis is not an instrument in the same way as F and U, but is a spread. Nevertheless we will say that we buy the basis when we buy U and sell F, and conversely.

Indeed, in many markets, the basis can be traded among operators when the transaction consists in instantaneously hedging U with F. Euronext lets available at the participants disposal a *basis trading* procedure for its Cac 40 and Notional Treasury Bond contracts, aimed at stimulating this basis business. The Exchange gives the following definition:

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*Post drawn from Echange de base, Matif SA (translated)*

A basis exchange operation consists in the simultaneous exchange between two operators of a given quantity of futures against a nominal amount of equivalent underlying assets.

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The basis is thus quoted:

$$[B_a : B_b] \text{ with } B_a = U_a - F_a \text{ and } B_b = U_b - F_b.$$  

In sum:

- the basis sells at $B_b$ — we sell U and buy F —
- the basis buys at $B_a$ — we buy U and sell F —

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5 "L’échange de base”, page 3, Matif SA, 1996.
The Management of Margin Calls Hypothesis

The future $F$ is supposed without daily margin call, as for a forward. It is therefore accompanied at expiration with a unique global margin call. This hypothesis will allow to set up pure arbitrages, i.e. perfectly without risk.

Another formulation of the hypothesis consists in saying that the margin calls are cumulated in a waiting account until the day of expiry, and settled only at that date. The account is not interest-bearing nor it is charged in case of outstanding negative balance. Interest on margin calls is thus left aside.

If this treatment does not strictly reflect the market practice\textsuperscript{6}, it is however a representation sufficiently close to it, and practical, so that researchers and Exchanges have adopted it in their works\textsuperscript{7} and presentations\textsuperscript{8}.

The price of a future with a unique global margin call at expiration is nonetheless not the price of a forward. The distinction stems from the difference between their transaction modes and from the right to prematurely close out a future pure arbitrage. This is the latter issue which is at the center of this paper\textsuperscript{9}.

II.2 Static Pure Arbitrage

II.2.1 Static Cash and Carry (C&C)

The cash and carry is one of the two pure arbitrages that are performed on the contracts. As its name suggests, it consists in buying with cash and carry an asset, in this case here asset $U$. By selling at the same time the contract at the price $F_b$, the arbitrageur contractually commits himself to exchanging this asset $U$ with the amount $F_b$, at expiry.

The arbitrage is said \textit{static} because the trader waits for the future maturing while doing nothing else.

- After the definition of pure arbitrage, the balance of the operation is worth zero at inception. Now if $U$ is bought spot, this balance is worth $-U_a$. One must then borrow to finance this purchase and bring back the initial balance to zero.

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\textsuperscript{6} In order to estimate the incidence of interest payments on margin calls, refer to chapter 5, “Arbitrage sur les marchés à terme”, Gilles Desvilles, PhD Dissertation, March 1998.


\textsuperscript{9} The incidence of the difference in transaction modes is presented in the article “Une nouvelle comparaison entre future et forward”, Gilles Desvilles, Banque et Marchés, 2000.
The prevailing borrowing rate is \( i_{t,a}^{T} \), which designates a ask quote, known at date \( t \) and spanning the period from \( t \) to \( T \). This is a spot actuarial annual rate that we will call more simply \( i_{a} \) in the absence of ambiguity.

The choice of an actuarial measure suits studying contracts of long dated maturities (we could have taken simple rates) and is in adequacy with the financial markets custom (we could have taken continuous rates).

The future trades at date \( t \) with the quotes \([F_{t,b}^{T} : F_{t,a}^{T}]\) written more simply \([F_{b} : F_{a}]\) in the absence of ambiguity.

Its daily settlement price at \( t \) is \( F_{set}^{t} \) and its last settlement price at expiry is \( F_{liq} \).

We name \( d \) the time to maturity of the contract: \( d_{t} = \frac{T(t) - t}{365} = T - t \) and count the dates in an annual basis while omitting the leap years to simplify the presentation. This time to maturity will be noted more simply \( d \).

Hence the representation:

```
\[ \begin{array}{c}
\text{i}_{a}, d \\
\text{t} \\
\text{U}_{a} \\
\text{F}_{b} \\
\text{T} \\
\text{F}_{liq} \\
\end{array} \]
```

**Laying the Cash Flows down Flat**

The course of the static cash and carry is the following:

<table>
<thead>
<tr>
<th>actions</th>
<th>flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>at ( t ): buys spot asset ( U )</td>
<td>-( U_{a} )</td>
</tr>
<tr>
<td>sells one future expiring at ( T ) for ( F_{b} )</td>
<td>0</td>
</tr>
<tr>
<td>borrows ( U_{a} ) at ( i_{a} )</td>
<td>( U_{a} )</td>
</tr>
<tr>
<td>balance at ( t )</td>
<td>( S_{t} = 0 )</td>
</tr>
</tbody>
</table>
At date $t$, $S_t$ is an already determined result, — the arbitrage payoff is a deterministic variable — . When at $t$ calculations bring out a positive balance $S_t$, the cash and carry arbitrage is initiated. Deciding on undertaking the arbitrage relies exclusively upon the result at expiration, since the intermediate balances are worth zero by construction.

**The Initial Margin Deposit**

The Exchanges require that each purchase or sale of a future be accompanied with an initial margin deposit, function of the contract nominal. Margin deposits in Treasury notes are accepted by the Exchanges without loss of accrued interest.

Margin deposits amount only from 0.2 to 5% of the nominal depending on the contracts. Moreover the arbitrageurs operate generally within large financial institutions who hold structurally very important amounts of Treasury notes.

So we will suppose in all the remaining that the arbitrageur owns the notes accepted by the Exchange in quality of margin deposit, deposits them as such, and fully benefits from their interest. As a consequence the initial margin deposit does not entail any cost and its presence is omitted.

**II.2.2 Static Reverse Cash and Carry (RCC)**

If calculations make a negative cash and carry payoff stand out, the arbitrage is not started. But then it becomes tempting to compute the ending balance of reverting the operations described above, so as to get the opposite of negative $S_t$, i.e. a credit balance. The strategy is called reverse cash and carry, and forms the second futures pure arbitrage.

Security lending is performed here without commissions. Taking account of these fees billed within the rates (all in) would be done easily by lowering the investment rate $i_b$.

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$^{10}$ The unique global margin call is the sum of the daily margin calls not reinvested nor refinanced, that is to say $(F_b - F_{\text{comp}}) + \sum_{t=2}^{T-1} (F_{\text{comp}}^{t-1} - F_{\text{comp}}^t) + (F_{\text{comp}}^{T-1} - F_{\text{liq}}) = F_b - F_{\text{liq}}$
The course of the static reverse cash and carry is the following:

<table>
<thead>
<tr>
<th>actions</th>
<th>flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>at t</strong></td>
<td></td>
</tr>
<tr>
<td>borrows asset U</td>
<td>0</td>
</tr>
<tr>
<td>sells asset U</td>
<td>$U_b$</td>
</tr>
<tr>
<td>buys one future expiring at $T$ for $F_a$</td>
<td>0</td>
</tr>
<tr>
<td>lends $U_b$ at $i_b$</td>
<td>-$U_b$</td>
</tr>
<tr>
<td>balance at $t$</td>
<td>$S_t = 0$</td>
</tr>
</tbody>
</table>

| **at $T_1$**                          |             |
| takes delivery of asset U             | 0           |
| returns asset U to security lender    | 0           |
| lets the contract expire, pays       | $-F_{liq}$  |
| honors the unique global margin call | $F_{liq} - F_a$ |
| retrieves the investment             |             |
| RCC balance at $T$                   | $S_T = -F_a + U_b (1 + i_b)^d$ |

When at $t$ calculations bring out a positive balance $S_T$, the RCC arbitrage is initiated.

**II.2.3 No Static Pure Arbitrage Opportunity Range (NSPAO)**

The No Static Pure Arbitrage Opportunity hypothesis, — a condition abbreviated here with NSPAO — , leads to the following delimitation:

$$U_b (1 + i_b)^d \leq F_b \leq F_a \leq U_a (1 + i_a)^d$$

The obtained bounds are the thresholds triggering the two pure arbitrages and form a classical result of the Cost of Carry Models\(^{11}\). Taking into account the market bid-ask spreads makes this kind of model more realistic.

\(^{11}\) "Understanding futures markets", pages 89 to 115, Robert W. Kolb, New York Institute of Finance, 3rd Edition. The Cost of Carry Models are amply exposed.
More interesting for what follows is to nearer future and forward: \( U_b (1 + i_b)^d \) is the forward price of \( U \), denoted \( F_{FWD}^b \), that a market-maker could ask should an OTC market exist for the future, and \( U_a (1 + i_a)^d \) the one he could offer, denoted \( F_{FWD}^a \).

\[
\begin{align*}
F_{FWD}^b &= U_b (1 + i_b)^d : F_{FWD}^a &= U_a (1 + i_a)^d \end{align*}
\]

is then named **forward bid-ask spread**.

In an efficient futures market, a market-maker does not generate any volume for by being executed within the forward bid-ask spread\(^{12}\) all the transactions escape from him. Nevertheless the banks stay in these markets for hedging and speculative motives, on proprietary or client accounts.

The future bid-ask spread \([F_b : F_a]\) fluctuates freely inside of the forward one \([F_{FWD}^b : F_{FWD}^a]\), as illustrated hereafter:

![Sketch 1](image)

Therefore \([F_{FWD}^b : F_{FWD}^a]\) is in itself the No Static Pure Arbitrage Opportunity range (NSPAO).

**II.3 Early Unwinding**

We insert now in our analysis the presettlement which represents a difference in the functioning of future and forward.

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\(^{12}\) See “Une nouvelle comparaison entre future et forward”, Gilles Desvilles, Banque et Marchés, 2000
II.3.1 Closing Out the Future, Offsetting the Forward

**Closing Out the Future**

When a participant sells a future, the Exchange registers an open position. When he buys it back subsequently, the Exchange closes this position. The contract is also closed: it gives rise to an instant payment of the difference between the buy price and the sell price, and the asset will need not be delivered anymore.

<table>
<thead>
<tr>
<th>Closing out and before term settlement of a future</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0$</td>
</tr>
<tr>
<td>Sale of F at $F_0^b$</td>
</tr>
<tr>
<td>Payment of $F_0^b - F_0^a$</td>
</tr>
</tbody>
</table>

**Offsetting the Forward**

By contrast, the forward is most of the time counterbalanced with a forward contracted in the reverse way, when one wants to stop its effect. Counterbalancing with a reverse forward is said offsetting the forward. Closing out as is done with futures is exceptional: this is called cancelling the forward, and comes very often along with expensive penalties.

The futures are often said fungible, and the forwards non-fungible.

The initial and the offsetting forwards are entered into the operator’s accounting and juridical books. Their expiration and settlement date is $T$.

<table>
<thead>
<tr>
<th>Offsetting and settling up a forward</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0$</td>
</tr>
<tr>
<td>Sale of F at $F_0^b$</td>
</tr>
<tr>
<td>Receipt of U, cash payment of $F_0^b$</td>
</tr>
<tr>
<td>Equivalent cash amount of $F_0^b - F_0^a$</td>
</tr>
</tbody>
</table>

The cash-flow in the future and forward cases is the same, $F_0^b - F_0^a$, but there shows up a lag in its payment.
\textbf{Cash Flow Lag}

We summarize this difference in functioning as follows:

<table>
<thead>
<tr>
<th>Cash flow lag</th>
<th>(t_0)</th>
<th>(t_1)</th>
<th>(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sale of (F)</td>
<td>—</td>
<td>—</td>
<td>(F_0^0 - F_0^1)</td>
</tr>
<tr>
<td>Purchase of (F)</td>
<td>—</td>
<td>(F_0^0 - F_0^1)</td>
<td>—</td>
</tr>
</tbody>
</table>

Forward

Future

The same lag is observed when buying first \(F\) and selling it after.

\textbf{II.3.2 Dynamic Cash and Carry}

We strive now at measuring the incidence of presettlement on futures pure arbitrages performed in a mature market.

\textbf{Cash Flows Table}

The arbitrageur does no longer hold his hedge until maturity \(T\) but unwinds it at \(t_1 < T\).

<table>
<thead>
<tr>
<th>C&amp;C unwound before term</th>
<th>(t_0)</th>
<th>(t_1)</th>
<th>(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>buys (U) at (U_0^0)</td>
<td>(-U_0^0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sells (F) at (F_0^0)</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>borrows spot (U_0^0) at (i_0^0)</td>
<td>(+U_0^0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sells (U) at (U_1^b)</td>
<td></td>
<td>(+U_1^b)</td>
<td></td>
</tr>
<tr>
<td>buys back (F) at (F_0^1)</td>
<td></td>
<td>(F_0^0 - F_0^1)</td>
<td></td>
</tr>
<tr>
<td>redeems borrowings early</td>
<td></td>
<td>(-U_0^0 \frac{(1+i_0^0)^d_0}{(1+i_0^b)^d_1})</td>
<td></td>
</tr>
<tr>
<td>Intermediate balances</td>
<td>(S_0 = 0)</td>
<td></td>
<td>(S_1)</td>
</tr>
<tr>
<td>Dynamic arbitrage payoff</td>
<td></td>
<td></td>
<td>(S_1)</td>
</tr>
</tbody>
</table>
Finally, the arbitrageur faces a unique balance $S_1$

$$S_1 = U^b_0 + F^0_b - F^0_a - U^a_0 \frac{(1+i^0_a)d_0}{(1+i^0_b)^d_1}$$

**Redeeming orOffsetting the Borrowings?**

The spot borrowings are refunded in advance by putting down

$$U^0_0 \frac{(1+i^0_a)d_0}{(1+i^0_b)^d_1}.$$

To understand where this amount comes from, we can suppose that the bank arbitrageur has issued a certificate of deposit at $t_0$ sufficiently liquid to be bought back in the secondary money market. This certificate is nothing else but a note $B(t_0)$ issued at the rate $i^0_0$.

In the markets reality, he will not be able to easily issue certificates whenever he needs, and the early redemption often comes along with penalties.

He will rather borrow $U^0_0$ at a fixed rate $i^0_0$ from $t_0$ to $T$ then lend $U^b_0$ at $i^b_0$ from $t_1$ to $T$. One often talks of offsetting the borrowings.

Offsetting brings in two cash flows:

$$F^0_b - F^0_a \text{ at } t_1 \quad \text{and} \quad U^b_0 (1+i^b_0)^d_1 - U^0_0 (1+i^0_0)^d_0 \text{ at } T$$

and so boils down almost to the same as redeeming early since the difference $U^b_0 (1+i^b_0)^d_1 - U^0_0 (1+i^0_0)^d_0$ is nothing but the future value at $T$, calculated at rate $i^b_0$, of:

$$U^b_0 - U^0_0 \frac{(1+i^0_a)d_0}{(1+i^0_b)^d_1}$$

Unfortunately the two cash flows occur at two different dates, $t_1$ and $T$. It is necessary then to take for example the future value at $T$ of the global margin call which can be either a debit or a credit: the capitalization rate is thus conditional upon the sign of the cumulated margins. This conditioning would make modelling as to follow more difficult (non recombining tree) probably without getting fundamentally different results. For sake of clarity we will keep studying with the initial borrowings early redemption hypothesis.

**II.3.3 Likely Existence of Opportunities**

Suppose an arbitrageur starts a dynamic cash and carry. We will show that there likely exists a future price such that unlatching early his hedge yields a profit whereas holding it till maturity causes a loss (consequently the future does not offer a static pure C&C opportunity ensuring that our discussion is adapted to all mature markets). Such a price must then verify a dynamic profit opportunity condition and the usual No Static Pure Arbitrage Opportunity (NSPAO) condition.
Dynamic Arbitrage Opportunity and NSPAO Conditions

Dynamic Arbitrage Opportunity Condition

There exists an early unwinding opportunity when there exists $t_1$ such that $S_1 > 0$, i.e.

$$U_b^i + F^0_b - F^1_a - U^0_a \frac{(1+i^0_a)^{d_0}}{(1+i^1_b)^{d_1}} > 0$$

NSPAO Condition

In order that the static C&C and RCC be non profitable, $F$ must fluctuate within the forward bid-ask spread as illustrated in sketch 1. In particular:

in $t_0$: $F^0_b \leq F_{wd,0}^a - \text{spd}^0$

in $t_1$: $F^1_a \geq F_{wd,1}^b + \text{spd}^1$

denoting $\text{spd}$ the future bid-ask spread — hence $F^1_a - F^0_b = \text{spd}^1$ — .

Example of Dynamic Arbitrage Opportunity

We analyse the scenario most favorable to the arbitrageur: he gets in by selling the future at the highest possible, i.e. at the C&C threshold, and gets out by buying it back at the lowest possible, i.e. at the RCC threshold. Therefore

$$F^0_b = F_{wd,0}^a - \text{spd}^0 = U^0_a (1+i^0_a)^{d_0} - \text{spd}^0$$

$$F^1_a = F_{wd,1}^b + \text{spd}^1 = U^1_b (1+i^1_b)^{d_1} + \text{spd}^1$$

By construction these values of $F^0_b$ and $F^1_a$ cannot be arbitraged statically and with them an opportunity to unwind early appears at $t_1$ whenever

$$S_1 = U_b^i + U^0_a (1+i^0_a)^{d_0} - \text{spd}^0 - U^1_b (1+i^1_b)^{d_1} + \text{spd}^1 - U^0_a \frac{(1+i^0_a)^{d_0}}{(1+i^1_b)^{d_1}} > 0$$

i.e.

$$U_b^i - U^0_a \frac{(1+i^0_a)^{d_0}}{(1+i^1_b)^{d_1}} < -\frac{\text{spd}^0 + \text{spd}^1}{(1+i^1_b)^{d_1-1}} < 0$$

A necessary condition (and very close to be sufficient so narrow the bid-ask spreads are in the major futures markets) is

$$\frac{U_b}{U^0_a} < \frac{(1+i^0_a)^{d_0}}{(1+i^1_b)^{d_1}}$$

$$\frac{\Delta U}{U} = \frac{U_b^i - U^0_a}{U^0_a} \frac{(1+i^0_a)^{d_0} - (1+i^1_b)^{d_1}}{(1+i^1_b)^{d_1}}$$
In quite many cases \((1+i_0^0)d_0\) is close to \((1+i_1^1)d_1\), or is even larger if the arbitrageur waits for a long time, and it suffices that the underlying \(U\) falls to create opportunities of winning early termination. Underlyings like an equity index and a notional bond are volatile enough to offer such exit windows.

The strict nature of the inequality \(S_1 > 0\) allows to assert that there exists in fact a bracket of values \(F^*_a\) for which unwinding is profitable:

\[
F^*_a > F_a > U_b^1 (1+i_b^1 d_1)
\]

where \(F^*_a\) is the break-even value at \(t_1\) of the dynamic arbitrage, i.e. it satisfies \(S_1 = 0\) which provides:

\[
F^*_a = U_b^1 + U_0^0 (1+i_0^0 d_0) - U_0^0 \frac{(1+i_0^0 d_0)}{(1+i_b^1 d_1)}
\]

Moreover, if \(spdU\) is the bid-ask spread of \(U\) — hence \(U_a^1 - U_b^1 = spdU^1\) — the condition so that the bracket of the winning exit values be larger than the NSPAO range is

\[
spdU < \frac{(1+i_0^0 d_0)}{(1+i_b^1 d_1)} \left( U_0^0 ((1+i_b^1 d_1) -1) - U_a^1 ((1+i_b^1 d_1) -1) \right)
\]

which obtains easily if the underlying has significantly dropped, which is consistent with the winning exit condition found above.

Moreover the fall of the underlying helps so much the dynamic C&C that the bracket of the winning exit values can even overlap the whole NSPAO range.

Returning to the scenario most in favor of the arbitrageur, we can illustrate it as follows:

Threshold to threshold fluctuation and unwinding of a C&C

Of course we do not claim that opportunities of getting out in advance of a dynamic C&C occur systematically. What we have just shown is that such opportunities arise in probability.
II.3.4 Dynamic Reverse Cash and Carry

Given the length of what has been said about the cash and carry, and its symmetry with respect to the reverse cash and carry, we will be more concise for the latter.

**Cash Flows Table**

<table>
<thead>
<tr>
<th>RCC unwound before term</th>
<th>( t_0 )</th>
<th>( t_1 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sells U at ( U^0_b )</td>
<td>+( U^0_b )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>buys F at ( F^0_a )</td>
<td></td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>lends spot(^{13}) ( U^0_b ) at ( i^0_b )</td>
<td>-( U^0_b )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| buys U at \( U^1_a \) | -\( U^1_b \) |         |         |
| sells back F at \( F^1_b \) | \( F^1_b - F^0_a \) |         |         |
| sells back the spot loan | +\( \frac{U^0_b (1+i^0_b)^d_0}{(1+i^0_a)^d_1} \) |         |         |

| Intermediate balances | \( S_0 = 0 \) | \( S_1 \) |         |
| Dynamic arbitrage payoff | \( S_1 \) |         |         |

With \( S_1 = -U^1_a + F^1_b - F^0_a + U^0_b \frac{(1+i^0_b)^d_0}{(1+i^0_a)^d_1} \)

**Example of Dynamic Arbitrage Opportunity**

We analyse here the scenario most favorable to the arbitrageur: he gets in by buying the future at the lowest possible, i.e. at the RCC threshold, and gets out by selling it back at the highest possible, i.e. at the C&C threshold. A necessary condition, and very close to be sufficient, of a money-making way out is

\[
S_1 = -U^1_a - U^0_b (1+i^0_b)^d_0 + U^0_b \frac{(1+i^0_b)^d_0}{(1+i^0_a)^d_1} + U^1_a (1+i^0_a)^d_1 > 0
\]

\(^{13}\) Equivalent to "buys a note B(.)" in order to avoid offsetting the loan. See discussion p 13.
which becomes: \[ \frac{U_a^0}{U_b^0} > \frac{(1+i_a^0) d_0}{(1+i_b^0) d_1} \]

Alike with C&C, we do not say that such winning exits appear systematically between \( t_0 \) and \( T \), but we remark that they can probably occur. The conjunction of a rise of the underlying — providing \( U_b^0 > U_b^0 \) — and of stable interest rates — providing \( i_a^0 \approx i_b^0 \) — is propitious to their coming out.

Threshold to threshold fluctuation and unwinding of a RCC

**II.4 Implicit Early Unwinding Options**

We just have seen that the relative fluctuations of \( F \) and \( U \), and in a lesser extent of \( i \), can provide the pure future arbitrageur with gain opportunities by undoing before term positions which lose money if held to expiration. In sum, the arbitrageur owns an implicit early unwinding option.

At any time \( t_1 < T \), the market value of a dynamic C&C is worth at least \( \max(S_1, VA(S_1)) \) where \( VA(S_1) \) is the present value of the C&C payoff known in advance if the hedge is held static. It can be worth more if prospects lead to expect superior gains \( S_1 \).

According to its definition the option is american and we now turn to express it formally, with a first step to identify its underlying.

**II.4.1 The Hedge as an Asset**

Hedge is a name universally known in both academic and operational finance, and generally interpreted as the combination of assets backing each other with a resulting risk needing control. It suits perfectly our analysis of dynamic arbitrages and we will use it to refer to the underlying of the option.
**Hedge**

The position generated by the C&C of the previous section is from now on considered as a composite asset. This asset is made up of an underlying U financed by issuing a debt B and immunized by selling a future F. We entitle it **hedge** and denote it H. So \( H = U - F - B \) is here an equivalence relationship between financial assets.

At any time the price of H is perfectly equal to the price of U, minus the cumulated margins of F, minus the price of B. It is nevertheless necessary to refine this equivalence owing to the buy-sell duality of quoted prices in financial markets.

**Buying and Selling Hedges**

Market participants have devised the concept of buying and selling hedges.

We will say here that the arbitrageur buys H, at \( H_a = U_a - F_b - B_a \), when he buys U, immunizes it with F and refinances it with B. This is a **buying hedge**. Likewise, he sells H at \( H_b = U_b - F_a - B_b + f_a \), when he sells U, invests the proceeds in B, and exposes it to U through F. This is a **selling hedge**.

Initiating a C&C amounts to purchasing a buying hedge and initiating a RCC amounts to purchasing a selling hedge. Untying prematurely a C&C boils down to sell the buying hedge, and a RCC to sell the selling hedge.

The purchase price of the buying hedge is worth zero since arbitrage is selffinanced. His sale price at t is exactly given by the C&C market value \( S_t \). Likewise the purchase price of the selling hedge is worth zero and his sale price is exactly given by the RCC market value.

**Fictitious Asset**

In view of what has just been said, the C&C unwinds at the sell price of the buying hedge now abbreviated with BH:

\[
BH_b^t = H_b^t = U_b^t + F_b^0 - F_a^1 - U_a^0 \frac{(1+i_b^0)}{(1+i_b^1)}
\]

and the RCC unwinds at the sell price of the selling hedge from now on written shortly SH:

\[
SH_b^t = -H_b^t = -U_b^t - F_a^0 + F_b^1 + U_b^0 \frac{(1+i_b^0)}{(1+i_b^1)}
\]

When the various bid-ask spreads are brought in, the asset hedge does not exist in full strictness since two distinct assets, BH and SH, are needed to value it. Nevertheless, for modelling convenience and in accordance to intuition, we will define it as a fictitious asset H issued at \( t_0 \) and whose quotes at any time t are \( [H_b^t : H_a^t] \) with by convention \( H_b^t = BH_b^t \) and \( H_a^t = -SH_b^t \).
Hence the **hedge bid-ask spread**:

\[
H_t^b = U_t^b + F_0^b - F_0 - U_0^b \frac{(1+i^b_0)^q}{(1+i^b_t)^q} \\
H_t^a = U_t^a + F_0^a - F_0 - U_0^a \frac{(1+i^a_0)^q}{(1+i^a_t)^q}
\]

\(H_t^b > H_t^a\) checks easily and supports that \(H_t^b\) and \(H_t^a\) are posited as the bid and the ask of the fictitious asset hedge.

Henceforward we will say equivalently that the arbitrageur unwinds his dynamic C&C or sells back his hedge, as well as unwinds his dynamic RCC or buys back his hedge.

**Graphical Interpretation**

The hedge quotes provide a very intuitive result. Starting as a C&C, the hedge is liquidated with profit as its bid is positive:

Starting in dynamic C&C at \(t_0\)

Starting in RCC, the hedge is sold with profit as its ask is negative.

**II.4.2 Hedge and Net Basis**

The hedge market value depends on the values of the underlying, the future and the interest rate. The future’s is narrowly linked to the underlying value since it lies between the bid and the ask of the forward quote stemming from the Cost of Carry model. However the
future displays a degree of freedom with respect to the model since it fluctuates, within this quote, wherever offer and demand carry it. The spread between the future price and the forward bid or ask is a contribution to the arbitrage performance as this shows up clearly in sketches 2 and 3. We express formally this spread in what follows.

**Breaking down the Hedge**

We force the bid and the ask of the forward quote to appear in the value of the hedge starting in C&C:

\[
H_b^1 = U_b^1 + F_b^0 - F_a^0 - U_a^0 \frac{(1+i_b^0)d_0}{(1+i_b^0)d_1}
\]

\[
= (F_{bFWD}^1 - F_b^0) - (F_{aFWD}^0 - F_b^0) + \left[U_b^1 - F_{bFWD}^1\right] + \left[F_{aFWD}^0 - U_a^0 \frac{(1+i_b^0)d_0}{(1+i_b^0)d_1}\right]
\]

The spread parting the future from the forward — \((F_{FWD} - F)\) — is called the **net of carry basis** by the futures traders, and is written here BN. Quite as for the basis seen in II.1.2, the presence of bid-ask spreads implies that the net basis has two values:

\[
(F_{bFWD}^1 - F_b^0) = BN_b \text{ is the net selling basis}
\]

\[
(F_{aFWD}^0 - F_b^0) = BN_a \text{ is the net buying basis}
\]

Moreover:

\[
U_b^1 - F_{bFWD}^1 = -U_b^1((1+i_b^1)d_1 - 1)
\]

\[
F_{aFWD}^0 - U_a^0 \frac{(1+i_b^0)d_0}{(1+i_b^0)d_1} = U_a^0 \frac{(1+i_b^0)d_0}{(1+i_b^0)d_1} \left((1+i_b^0)d_1 - 1\right)
\]

Hence

\[
H_b^1 = BN_b - BN_a + \left[U_a^0 \frac{(1+i_b^0)d_0}{(1+i_b^0)d_1} - U_b^1\right] \left((1+i_b^0)d_1 - 1\right)
\]

Therefore the hedge value is split into two kinds of component:

- a **within NSPAO range fluctuation component**: \(BN_b - BN_a\)

- a **market fluctuation component**: \(\left[U_a^0 \frac{(1+i_b^0)d_0}{(1+i_b^0)d_1} - U_b^1\right] \left((1+i_b^0)d_1 - 1\right)\)

The market fluctuation component brings in both the underlying and the short rates but it is clear that the influence of a currency, equity or bond underlying is there overwhelming. The only exception is the case where the underlying is itself a short rate asset.
**Net Basis**

The subsequent breaking down is useful to shed more light on these concepts:

\[
\begin{align*}
B_a &= \text{buying basis} = U_a - F_b \\
BN_a &= \text{net buying basis} \\
\text{buying carry} &= -U_a [(1+i_a)^d-1] \\
B_a &= \text{buying carry} + BN_a
\end{align*}
\]

The net buying basis is by definition always positive in a market where NSPAO prevails, whether the basis and its carry be positive or negative — the negative case being presented above — .

\[
\begin{align*}
B_b &= \text{selling basis} = U_b - F_a \\
BN_b &= \text{net selling basis} \\
\text{selling carry} &= -U_b [(1+i_b)^d-1] \\
B_b &= \text{selling carry} + BN_b
\end{align*}
\]

The net selling basis is by definition always negative in a NSPAO market, whatever the sign of the basis and its carry. In the above sketch they show up also negative but this representation is not intended to be generalized.

Net buying and selling bases wrap the NSPAO range, a feature characterized by the following identity:

\[
\text{NSPAO range width} = BN_a - BN_b - (F_a - F_b)
\]
**Synoptic Table**

We gather the dynamic pure arbitrage payoffs in the following table:

<table>
<thead>
<tr>
<th>Synoptic Table</th>
<th>Dynamic Pure Arbitrage Payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>C&amp;C start</strong></td>
</tr>
<tr>
<td>Unwinding at $t_1$</td>
<td>$H^c_b = BN^c_b - BN^c_T + \left[ U^c_0 \frac{(1+i^c_0)^d_0}{(1+i^c_b)^d_1} - U^c_1 \right] ((1+i^c_b)^d_1 - 1)$</td>
</tr>
<tr>
<td></td>
<td><strong>RCC start</strong></td>
</tr>
<tr>
<td>Unwinding at $t_1$</td>
<td>$-H^r_a = -BN^r_a + BN^r_T + \left[ U^r_0 - U^r_1 \frac{(1+i^r_0)^d_0}{(1+i^r_a)^d_1} \right] ((1+i^r_a)^d_1 - 1)$</td>
</tr>
</tbody>
</table>

In the two cases there are two different sources contributing to the hedge value:

1. **a within NSPAO range fluctuation** component:

   $BN^c_b - BN^c_T$ for the cash and carry starts, $-BN^r_a + BN^r_T$ for the reverse ones.

2. **a market fluctuation** component:

   $\left[ U^c_0 \frac{(1+i^c_0)^d_0}{(1+i^c_b)^d_1} - U^c_1 \right] ((1+i^c_b)^d_1 - 1)$ for the C&C, $\left[ U^r_0 - U^r_1 \frac{(1+i^r_0)^d_0}{(1+i^r_a)^d_1} \right] ((1+i^r_a)^d_1 - 1)$ for the RCC.

At this stage it is important to notice that the within NSPAO range fluctuation component is always negative for the net buying and selling bases are respectively always positive and negative. Its contribution to dynamic arbitrage is thus negative. Say it another way, the arbitrageur cannot take advantage of the net basis fluctuations, quite on the contrary he only derives losses from them.

In corollary the basis trading, whom one shows that it necessarily boils down to trading the net basis, fundamentally loses money, excepted by trying to catch up this structural handicap through speculation in the market fluctuation component.

**II.4.3 Call and Put on the Hedge**

In a first time, in order to help understanding, we let the arbitrages start from the two NSPAO range boundaries, then we will relax this hypothesis.
**Call on the Hedge**

We suppose that at $t_0$ the arbitrageur triggers off a C&C by selling a future at exactly the static pure arbitrage threshold, i.e. at $F_{0}^{0} = F_{0}^{	ext{FWD,0}}$. Thus he enters a buying hedge BH which, if held to maturity, will return nothing to him ($S_T = 0$). But if at date $t_1$ he jumps at the opportunity to sell BH as it is positive — and such an opportunity can arise as demonstrated in II.3.3 and II.3.4 —, the C&C unwound before term turns out to be profitable. On the contrary if the arbitrageur notes that BH remains always negative then he never sells back the hedge and ends up at $T$ with a zero payoff.

This position where at all time one gains under condition and never loses, and which stops at $T$, is therefore worth something. Its value is the price of an *american option* expiring at $T$. Furthermore, as it has been agreed that $BH_b = H_b$, i.e. equal the bid of the hedge taken as an asset, it is thus the hedge bid which makes up the option underlying. The option then likens a call on the hedge with a zero strike — for with a call one buys at strike and sells back at bid —.

**Put on the Hedge**

What we have just analysed about the dynamic C&C and its call applies as well to the RCC in return for a few simple adjustments.

At $t_0$ an arbitrageur triggers off a RCC by buying a future at the static pure arbitrage threshold $F_{0}^{0} = F_{0}^{	ext{FWD,0}}$. He then enters a selling hedge SH which, if he holds it to maturity, will not cost nor pay him anything. But if at date $t_1$ he takes the opportunity to sell SH as $SH_{t_1} > 0$, then he will make money.

Now by our convention $SH_{t_1} = -H_{t_1}$. So the arbitrageur benefits from an american put on the hedge $H$ with a zero strike.

**Generalization**

If the arbitrageur sets up a static cash and carry as the future lies strictly inside of the NSPAO range, then by definition he will lose money at expiration. But if the future is not too far from the NSPAO range upper boundary, he can reasonably expect to be able in the meantime to unwind his position at a gain.

The same option characteristics subsist, only the terminal condition is modified. In the previous specific case the hedge could be held till expiration yielding a zero terminal cash flow, now the latter is negative. The adjustment is thus tiny and does not call back into question the existence of the *unwinding option* that we have just expressed formally.
The hedge departs neatly from underlyings modelled by Black and Scholes or Cox, Ross and Rubinstein\textsuperscript{14}. Indeed $H_b$ does not follow a log-normal law with constant volatility because of the hedge convergence feature when nearing maturity, but as well for an empirical reason: $H_b$ can follow trends varying under the effect of the offer and demand for futures. Hence calculating the value of the unwinding option will not be classical.

### III Numerical Valuation of the Early Unwinding Option

This part values an unwinding option in a dynamic cash and carry. In the synoptic table in II.4.2 the hedge appears as a function of the underlying, the interest rate and the net basis:

$$H_b^t = B N_b^t - B N_a^t + \left[ U_0 \frac{(1+i_b^t)^d_0}{(1+i_b^t)^d_1} - U_b^t \right] \left( (1+i_b^t)^d_1 - 1 \right)$$

In this paper we assume that the interest rates are constant\textsuperscript{15}, it will no longer be necessary to timestamp them. The option model here is thus a **two factor model**: the factors are the future’s underlying asset and the net basis.

Unlike Brennan and Schwartz, we will not let the hedge diffuse according to a process taken without justification from a handbook of Statistics but will use its above expression, i.e. a function of the underlying and of the net basis, which they, on the other hand, will be modelled in a way generally accepted by the financial markets: the underlying will evolve on a binomial tree à la Cox, Ross and Rubinstein and will asymptotically follow a log-normal law, and the net basis will swing from its minimum to its maximum. This approach warrants that the hedge takes **endogeneous** values with respect to the realistic market that we have represented here, i.e. with bid-ask spreads and no free lunch from static pure arbitrages.

Finally we will price the early unwinding option by putting ourselves in a risk-free world.

---


\textsuperscript{15} The model is fundamentally a three factor one and its representation would be an octonomial tree. The constant interest rate hypothesis removes one factor.
III.1 Factors Diffusion

III.1.1 Factors Tree

The Binomial Tree of the Future Underlying Asset

The underlying is quoted \([U_b - U_a]\) but it is the mid-quote \(U\) which evolves on a binomial tree, \(U_b\) and \(U_a\) being deduced from \(U\) by applying a proportional bid-ask spread rate \(f\):

\[ U_b = U (1 - f) \quad \text{and} \quad U_a = U (1 + f). \]

The mid is constrained either to geometrically grow at the rate \(u\) or decrease at the rate \(d\) over one period, which gives the following classical tree, developed over two \(\Delta t\) periods:

\[
\begin{align*}
\begin{array}{c}
\text{\(t_0 = 0\)} \\
\text{\(t_1 = t_0 + \Delta t\)} \\
\text{\(t_2 = t_1 + \Delta t\)}
\end{array}
\end{align*}
\]

\[
\begin{align*}
U & \\
UU & = U (1+u)^2 \\
Ud & = U (1+u) \\
Ud & = U (1-d) \\
Udd & = U (1-d)^2
\end{align*}
\]

The tree values of a random variable like \(U\) will be called allowed values in this paper.

The trees of the bid and the ask of \(U\) are deduced from the above by applying \(1 - f\) and \(1 + f\) to every node. The three trees recombine.

If the underlying mid has an observed volatility \(\sigma\) we choose the following values for \(u\) and \(d\)\(^{16}\), so that the volatility of the binomial process is also \(\sigma\):\n
\[
\begin{align*}
u &= e^{\sigma \sqrt{\Delta t}} - 1 \quad \text{and} \quad d = 1 - e^{-\sigma \sqrt{\Delta t}}
\end{align*}
\]

\(u\) and \(d\) are independent from the probabilities of rise and fall.

The Quadrinomial Tree of the Net Basis

The net basis is not an asset nor a derivative, it measures the gap of the future to the two no-static-pure-arbitrage thresholds. Furthermore this spread is bounded at any time, whereas asymptotically the underlying price is not for it follows a log-normal law.

These remarks preclude the binomial tree with geometric growth model. Nevertheless the arborescence principle is kept in order to get a joint diffusion of the two factors, which allows calculations of reasonable complexity.

---

\(^{16}\) Following in this Cox, Ross and Rubinstein, "Option Pricing: A Simplified Approach", Journal of Financial Economics, 1979, but these are not the only possible values for \(u\) and \(d\). These values imply that \(u = 1/d\).
The up and downs of the net basis stem from the interplay between offer and demand for the future. As seen in II.4.2, owing to the component \( BN_a \) – \( BN_0 \) they contribute to the dynamic cash and carry performance from \( t_0 \) to \( t_1 \).

For two given dates the most favorable case is the one illustrated in sketch 2. In a general way, the contribution is all the better as the net buying basis \( BN_a \) and the net selling basis \( BN_b \) (in absolute value) are small, which reflects that one should rather initiate a dynamic C&C when the future is rich and unwind it when it is cheap.

By definition:
\[
F^{FWD}_a - F^{FWD}_b \geq BN_a \geq F_a - F_b
\]
\[
-(F_a - F_b) \geq BN_b \geq -(F^{FWD}_a - F^{FWD}_b)
\]

\( F_a - F_b \) is the future bid-ask spread and \( F^{FWD}_a - F^{FWD}_b \) the NSPAO range width, and they are denoted respectively \( s \) and \( l \) in this third part. We suppose that \( s \) is the same at any time, which mirrors the major futures markets reality. On the other hand \( l \) varies with both the time and the underlying, and it will be proper to index it with these two variables.

From \( t_0 \) to \( t_1 \) the case most favorable to dynamic cash and carry is thus:
\[
BN_0 = s \quad \text{and} \quad BN_1 = -s
\]
and the most unfavorable:
\[
BN_0 = l^0 \quad \text{and} \quad BN_1 = -l^1
\]

\( BN_0 \) being the tree starting point it must be frozen at an arbitrary value that can be a parameter. In the model \( BN_1 \) can take one of its two extreme values, \(-s\) or \(-l\). However as said before the NSPAO range width depends on the time to expiration and the underlying level. For instance, after the underlying rises twice in a row, the width is worth:
\[
l_{uu} = F^{FWD}_{uu,a} - F^{FWD}_{uu,b} = U_{uu} (1+f) (1+i_a)^d_2 - U_{uu} (1-f) (1+i_b)^d_2
\]
\[
= U (1+u)^2 [(1+f)(1+i_a)^d_2 - (1-f)(1+i_b)^d_2]
\]
and after a rise and fall:
\[
l_{ud} = U (1+u)(1-d) [(1+f)(1+i_a)^d_2 - (1-f)(1+i_b)^d_2]
\]
Hence the recombining widths tree:

\[
\begin{array}{c|c|c}
  t_0 = 0 & t_0 + \Delta t & t_0 + 2\Delta t \\
  \hline
  \text{l}^0 & \text{l}_u & \text{l}_{uu} \\
  \text{l}_d & \text{l}_{ud} & \text{l}_{dd}
\end{array}
\]

Width depending on $U$ makes the net selling basis tree quadrinomial, yet this net basis can settle in only two states, its minimum and its maximum:

\[
\begin{array}{c|c|c}
  t_0 = 0 & t_0 + \Delta t & t_0 + 2\Delta t \\
  \hline
  -\text{l}_{uu} & -\text{s (up up)} & -\text{s (up down)} \\
  -\text{l}_u & -\text{s (up)} & -\text{l}_{ud} \\
  -\text{l}_d & -\text{s (down up)} & -\text{s (down down)} \\
  -\text{s (down)} & -\text{s (down)} & -\text{s (down)}
\end{array}
\]

Recombinations belonging to the model prevent the nodes from growing as $4^n$ with $n$ being the periods number, which would slow down the simulation to be presented farther in this part. At period $n$ there are only $2(n+1)$ nodes.
III.1.2 Valuation in a Risk-Neutral World

**Underlying Risk-Neutral Probability**

The risk-neutral valuation technique has become classical particularly with a geometric growth binomial tree\(^\text{17}\). The mathematical conditions for using it are here filled (measurable space, adapted filtration).

In the standard case of \( U \) mid, the probability measure in a risk-neutral world exists, is unique and is characterized by the probability of an upmove \( \pi \):

\[
\pi = \frac{(1+i)^{\Delta t} - (1-d)}{u-d} = \frac{(1+i)^{\Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} + e^{-\sigma \sqrt{\Delta t}}}
\]

where \((1+i)^{\Delta t}\) acts here as the rate of the new numéraire, the money market account.

With this probability measure the mid of any derivative \( V \) on \( U \) obtains through the risk-neutral valuation equation:

\[
V_t = \mathbb{E}_Q \left[ V_{t+\Delta t} \right]
\]

where \( \mathbb{E}_Q[ \cdot ] \) represents the expectation operator in the risk-neutral world probabilized with \( \pi \).

**Net Basis Risk-Neutral Probability**

The framework of the net basis is less standard for on one hand it is not an asset and on the other hand its growth is not geometric.

The first hurdle is lifted by representing the states of the net basis in the previous chart with the mid of the future, which, for its part, is an asset. As for the underlying this is the mid of the future, denoted \( F \), that we let diffuse:

\[
\begin{align*}
F_0,0,0 & \\
\pi \lambda, F_0 & \\
\pi (1-\lambda), F_0 & \\
(1-\pi) \lambda, F_0 & \\
(1-\pi)(1-\lambda), F_0 & \\
\end{align*}
\]

\( \lambda \) is the risk-neutral probability that the the net selling basis be maximal in absolute value, that is to say equal to the width \( I \) of the NSPAO range.

---

\(^{17}\) See for example Steven Schreve, “Stochastic Calculus and Finance”, draft, 1997, chapter 3.
As the future mid reaches the NSPAO range upper boundary, the ask is above by a half-spread $s/2$, contrary to what show the sketches 2 and 3. The same remark applies to the bid and the lower boundary. This slight contradiction bears no theoretical nor practical importance for whether the future ask be above the forward ask or the future bid be below the forward bid does not offer static pure arbitrage opportunities, and furthermore $s/2$ is very small in the major futures markets.

The market value of the future, not to be confused with the future price, is the value of its margin call\(^{18}\). Hence the value tree:

\[
\begin{align*}
0 & \quad \pi \lambda \\
& \quad \pi (1-\lambda) \\
& \quad (1-\pi) \lambda \\
& \quad (1-\pi)(1-\lambda) \\
F_{a,u}^{\text{FWD}} - F^0 & \quad \pi \lambda \\
F_{b,u}^{\text{FWD}} - F^0 & \quad \pi (1-\lambda) \\
F_{a,d}^{\text{FWD}} - F^0 & \quad (1-\pi) \lambda \\
F_{b,d}^{\text{FWD}} - F^0 & \quad (1-\pi)(1-\lambda)
\end{align*}
\]

Applying the risk-neutral valuation equation leads to:

\[
0 \times (1+i)^\Delta t = \pi \lambda (F_{a,u}^{\text{FWD}} - F^0) + \pi (1-\lambda) (F_{b,u}^{\text{FWD}} - F^0) + (1-\pi) \lambda (F_{a,d}^{\text{FWD}} - F^0) + (1-\pi)(1-\lambda) (F_{b,d}^{\text{FWD}} - F^0)
\]

hence

\[
\lambda = \frac{F^0 - (\pi F_{b,u}^{\text{FWD}} + (1-\pi) F_{b,d}^{\text{FWD}})}{\pi U u + (1-\pi) U d}
\]

This expression provides a value of $\lambda$ contained between 0 and 1 when $F^0$ lies enough inside of the NSPAO range. Furthermore $\lambda$ witnesses an interesting property: it increases from about zero to about one as $F^0$ goes from $F_{b}^{FWD,0}$ to $F_{a}^{FWD,0}$. Unfortunately if $F^0$ is very close to the lower boundary $F_{b}^{FWD,0}$ then $\lambda$ becomes slightly negative. Indeed, this can be seen by taking the extreme but allowed case $F^0 = F_{b}^{FWD,0}$, and noticing that ($T$ is the future expiry date)

\[
\pi F_{b,u}^{\text{FWD}} + (1-\pi) F_{b,d}^{\text{FWD}} = \pi U u (1-t) (1+i_b) T^-\Delta t + (1-\pi) U d (1-t) (1+i_b) T^-\Delta t
\]

\[
= (\pi U u + (1-\pi) U d) (1-t) (1+i_b) T^-\Delta t = F_{b}^{FWD,0}
\]

\[
= \left(1+i \left(1+i_b\right)^\Delta t\right)^{\Delta t} F_{b}^{FWD,0}
\]

\[^{18}\text{See for example Jarrow and Turnbull, "Derivative Securities", 2nd edition, South-Western, 2000, chapter 6 paragraph 3, for this approach.}\]
Likewise when $F_0$ nears the upper boundary $F_{a,d}^{\text{FWD,0}}$, $\lambda$ slightly exceeds one, as one sees by taking the extreme but allowed case $F_0 = F_{a,d}^{\text{FWD,0}}$ and noticing that

$$
\pi F_{a,u}^{\text{FWD}} + (1-\pi) F_{a,d}^{\text{FWD}} = \left(1 + \frac{i}{1+i_a}\right) F_{a,d}^{\text{FWD,0}}
$$

and

$$
\pi u + (1-\pi) d = \pi F_{a,u}^{\text{FWD}} + (1-\pi) F_{a,d}^{\text{FWD}} - (\pi F_{d,u}^{\text{FWD}} + (1-\pi) F_{d,d}^{\text{FWD}}) = \left(1 + \frac{i}{1+i_a}\right) F_{a,d}^{\text{FWD,0}} - \left(1 + \frac{i}{1+i_b}\right) F_{b,d}^{\text{FWD,0}}.
$$

Strictly speaking it is thus impossible to extract from the net basis tree risk-neutral probabilities always contained between zero and one. Yet risk-neutral valuation offers almost the property that when the future reaches a boundary then afterwards it sticks to it, which makes the quadrinomial tree equivalent to two classical binomial subtrees. The absence of interest rate bid and ask would make this property perfectly true. In order to integrally safeguard the transaction costs, use the risk-neutral valuation which is a financial markets standard, and benefit from the breaking down into two binomial subtrees, we slightly restrict the range where the future mid price freely fluctuates in the following way:

$F_0$

- $\pi \lambda$
- $\pi (1-\lambda)$
- $(1-\pi) \lambda$
- $(1-\pi)(1-\lambda)$

is called **adjusted NSPAO range**.

The new allowed values of $F$ are not far from the true static pure arbitrage boundaries if the interest rate bid-ask spread is narrow as it is the case in mature markets.

With this restriction calculus brings

$$
\lambda = \frac{F^0 - \left[ \pi \left(1 + \frac{i}{1+i_a}\right) F_{a,d}^{\text{FWD}} + (1-\pi) \left(1 + \frac{i}{1+i_b}\right) F_{b,d}^{\text{FWD}} \right]}{\pi \left(1 + \frac{i}{1+i_a}\right) F_{a,u}^{\text{FWD,0}} - \left(1 + \frac{i}{1+i_b}\right) F_{b,u}^{\text{FWD,0}} + (1-\pi) \left(1 + \frac{i}{1+i_a}\right) F_{a,d}^{\text{FWD,0}} - \left(1 + \frac{i}{1+i_b}\right) F_{b,d}^{\text{FWD,0}}}
$$

$$
\lambda = \frac{F^0 - F_{b,d}^{\text{FWD,0}} \left(1 + \frac{i}{1+i_b}\right)^{2\Delta t}}{F_{a,d}^{\text{FWD,0}} \left(1 + \frac{i}{1+i_a}\right)^{2\Delta t} - F_{b,d}^{\text{FWD,0}} \left(1 + \frac{i}{1+i_b}\right)^{2\Delta t}}
$$
This expression gives a value of $\lambda$ contained between zero and one as $F$ lies inside of the
adjusted NSPAO range$^{19}$: a risk-neutral probability so exists, is unique and consequently our
model precludes all riskfree arbitrage and its market framework is complete$^{20}$. Moreover $\lambda$
is exactly worth zero when $F$ is at the lower adjusted boundary $(1+i/1+i_b)^{2\Delta t} F^{0,\text{FWD}}_b$ and one
when $F$ is at the upper adjusted boundary $(1+i/1+i_a)^{2\Delta t} F^{\text{FWD,0}}_a$, which implies that in a risk-free
world the net basis gives up all its volatility and, after one period, remains always at either
its high or its low level$^{21}$.

This latter property will simplify the unwinding option valuation because the hedge
quadrinomial tree will amount to two recombining binomial subtrees.

However we leave open to the discussion to which extent the boundary adjustment
introduces a downward bias in the option price. Indeed preventing the future to reach
the most favorable upper boundary forbids some outcomes that the american feature of the
option could exploit.

III.2 Underlying and Option Diffusion

III.2.1 Hedge Tree

The quadrinomial feature of the net selling basis tree is passed along to the hedge tree
since the hedge market value brings in the value of the net selling basis:

$$H_b^i = B N_b^i - B N_b^0 + \left[U_b^0 \frac{(1+i a)^{\Delta t}}{a} - U_b^1 \right] \left( (1+i_b)^{\Delta t} - 1 \right)$$

First Period Quadrinomial Tree

Prior to calculating $H_b$ it is proper to let diffuse the future mid with its two adjustment
coefficients $(1+i/1+i_b)^{\Delta t}$ and $(1+i/1+i_a)^{\Delta t}$ as seen above, then to deduce from this the
diffusion of the net selling basis. So this adjusted diffusion differs from the one presented in
III.1.1., but it remains very close to it as $\Delta t$ gets small and for this reason we index the net

---

19 Two alternative processes avoiding this adjustment are exposed in the Appendix but they make
considerably more cumbersome the formal representation of the option and its numerical valuation.

20 Results owed to Harrison and Pliska in “Martingales and Stochastic integrals in the theory of continuous
trading, Stochastic Processus and Applications”, 1981, pp 215-260 and in “A stochastic calculus model of

21 These relationships at $2\Delta t$ define recursively the adjusted boundaries. Besides $\lambda = 0$ or 1 is not an
assumption but a result of the risk-neutral valuation. In the real world $\lambda$ can take quite another value.
basis with the unadjusted terms \( s \) and \( l \). For instance, after a rising \( U \) over the first period and with \( F \) at the higher level in the NSPAO range, \( BN_b \) is written \( lu BN_b \) and is worth

\[
lu BN_b = F_{FWD}^u - \left( \frac{1 + i}{1 + i_0} \right)^{\Delta t} F_{FWD}^a - s/2 \approx F_{FWD}^u - F_{FWD}^a = -lu
\]

and with \( F \) at the lower level, it is written \( s u BN_b \) and is worth

\[
s u BN_b = F_{FWD}^u - \left( \frac{1 + i}{1 + i_0} \right)^{\Delta t} F_{FWD}^a - s/2 \approx -s
\]

In the first period the hedge tree is quadrinomial for the probability of reaching the adjusted C&C boundary is a priori different from zero and one. Indeed the risk-neutral value to the future mid at \( t_0 \) is equal to its real world price, and this price has no reason to be equal to one of the two adjusted NSPAO boundaries. Let \( \lambda_0 \) be the probability at \( t_0 \).

The hedge takes on four possible values according to whether the underlying rises or falls and whether the adjusted net selling basis is maximal in absolute value (i.e. worth about the NSPAO range width, \( l \)) or minimal (about the future bid-ask spread, \( s \)).

<table>
<thead>
<tr>
<th>( t_0 = 0 )</th>
<th>( t_0 + \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s H_0' ) = ( s u BN_b - BN_b + \left[ \frac{U_0^u (1 + i_0)^T}{(1 + i_b)^T} - \frac{U_0^a (1 + u)^T}{(1 + i_b)^T - 1} \right] )</td>
<td>( \pi \lambda_0 )</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>( lu H_0' = )</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>( \pi (1 - \lambda_0) )</td>
</tr>
<tr>
<td>(1 - ( \pi )) ( \lambda_0 )</td>
<td>( s H_0^d = )</td>
</tr>
<tr>
<td>( s H_0^d = s BN_b - BN_b + \left[ \frac{U_0^u (1 + i_0)^T}{(1 + i_b)^T} - \frac{U_0^a (1 + u)^T}{(1 + i_b)^T - 1} \right] )</td>
<td>(1 - ( \pi )) ( (1 - \lambda_0) )</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>( ld H_0^d = )</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>( ld BN_b - BN_b + \left[ \frac{U_0^u (1 + i_0)^T}{(1 + i_b)^T} - \frac{U_0^a (1 + u)^T}{(1 + i_b)^T - 1} \right] )</td>
</tr>
<tr>
<td>( \lambda_0 )</td>
<td>( (1 - \lambda_0) )</td>
</tr>
</tbody>
</table>

**Subsequent Periods Binomial Subtrees**

In the second period the hedge restarts from each of the four above values with the same kind of tree as in the first period. This tree is thus a priori quadrinomial. However we have obtained in III.1.2. that \( \lambda_1 \) becomes identically zero or unity thanks to the tiny adjustment of the future allowed values. The consequence of this result reads in the above drawing, while replacing \( \lambda_0 \) with \( \lambda_1 \): if \( \lambda_1 \) is worth zero then the branches
probabilized with $\pi \lambda$ and $(1-\pi)\lambda$ vanish, and if $\lambda$ is worth one these are the two others that wipe out.

Let us put ourselves at $t_0 + \Delta t$ with a future at the adjusted upper boundary: the probability that it be there too one period later is one. This implies that from this node only two branches ramify. As the underlying admits two states, there are at $t_0 + \Delta t$ two nodes at the upper boundary (the first and the third on the above diagram), from each of which only two branches ramify, a priori yielding two times two equals four new nodes at $t_0 + 2\Delta t$.

However elementary formal calculus shows that two out of the four branches recombine into one same node, which brings down the number of new nodes from four to three:

\[
\begin{array}{c|c}
\text{t0 + Δt} & \text{t0 + 2Δt} \\
\hline
\end{array}
\]

\[
luuH^u = luuBN^u - BN^d + [u_a^p (1+i_b)^{T-2\Delta t} - u_a^l (1+u)^2]((1+i_b)^{T-2\Delta t-1})
\]

\[
ludH^d = ludBN^d - BN^d + [u_a^p (1+i_b)^{T-2\Delta t} - u_a^l (1+u)(1-d)]((1+i_b)^{T-2\Delta t-1})
\]

\[
luddH^d = luddBN^d - BN^d + [u_a^p (1+i_b)^{T-2\Delta t} - u_a^l (1-d)^2]((1+i_b)^{T-2\Delta t-1})
\]

As the scenario repeats with $\lambda_2 = \lambda_3 = ... = \lambda = 1$, the hedge subtree starting at $t_0 + \Delta t$ from the nodes where the future lies on the adjusted upper boundary is a recombining binomial tree. At date $t_0 + n\Delta t$ it has $n+1$ nodes.

Likewise the subtree starting at $t_0 + \Delta t$ from nodes where the future is at the adjusted lower boundary is also binomial and recombining.

Therefore pricing the unwinding option whose underlying is the hedge boils down to pricing two options on an asset evolving on a recombining binomial tree.

**III.2.2 Option Trees**

The dynamic C&C unwinding option is a call on the hedge. We adopt the following notations, similar to those of the net selling basis:

- The call value is $luCu$ as the underlying $U$ rises by $u$ and the net selling basis $BN_b$ is worth about $-lu$, $luCuuu$ as $U$ rises twice by $u$ and $BN_b$ is worth about $-luu$, and so forth.
The call value is \( sCu \) as the underlying \( U \) rises by \( u \) and the net selling basis \( BN_b \) is worth about \(-s\), \( sCuu \) as \( U \) rises twice by \( u \) and \( BN_b \) is worth about \(-s\), and so forth.

After the first period \( \Delta t \) the call has probabilities \( \lambda_0 \) and \( 1 - \lambda_0 \) to take two disjoint subtrees and can so be broken down into two sub-calls named \( lC^0 \) and \( sC^0 \):

| Call subtree as the future nears the upper boundary of the NSPAO range \( |BN_b| \approx l \) | Call subtree as the future nears the lower boundary of the NSPAO range \( |BN_b| \approx s \) |
|---|---|
| \( t_0 = 0 \) | \( t_0 = 0 \) |
| \( t_1 = t_0 + \Delta t \) | \( t_1 = t_0 + \Delta t \) |
| \( t_2 = t_0 + 2\Delta t \) | \( t_2 = t_0 + 2\Delta t \) |

\[\begin{array}{c|c|c|c}
\pi & luC & luuCuu \\
\hline
1-\pi & lCd & ludCud \\
\end{array}\]

\[\begin{array}{c|c|c|c}
\pi & sC & sCuu \\
\hline
1-\pi & sCd & sCd \\
\end{array}\]

**Valuation using the Dynamic Programming Equation**

The call on the hedge is American and its valuation uses the dynamic programming principle\(^{22}\), which consists in comparing the value of exercising now with the expected discounted one period ahead option value.

Suppose that the future expires at the end of two periods: \( T = t_2 = t_0 + 2\Delta t \).

The intermediate call value after the underlying moves up and the future is rich is:

\[ luC = \max [ luBH^u, (1+i)^{-\Delta t}(\pi luCuu + (1-\pi) ludCud) ] \]

We discount with the mid-interest rate for the risk-neutral world is probabilized with \( \pi \) itself being established with the aid of a mid rate.

The terminal value of the call after the underlying rises twice and with a rich future is:

\[ luuCuu = \max ( luuBH^u, -BN^u ) = \max ( luuBN^uu - BN^u, -BN^u ) = -BN^u \]

for the net selling basis is always negative as NSPAO prevails.

The same argument applies to all terminal values which are so equal to \(-BN^u\).

\(-BN^u\) is what it costs by holding the dynamic C&C until the expiry date, for at that time the arbitrageur is better off not offsetting his position but rather letting it expire in order to save on transaction costs.

The call current price is the average at \( t_0 \) of the two prices got on the subtrees:

---

\(^{22}\) See for example Steven Schreve, "Stochastic Calculus for Finance I", Springer, 2005, chapter 4, theorem 4.4.3.
\[ C^0 = \lambda_0 lC^0 + (1-\lambda_0) sC^0 \]

### III.3 Numerical Valuation of the Early Unwinding Option

#### III.3.1 Valuation without Transaction Cost

In the absence of transaction costs the NSPAO range does not exist any longer and along with it so do the net basis fluctuations, consequently the market value of the hedge opened at \( t_0 \) and closed out at \( t_1 \) amounts to

\[ H_b = \left[ U^0 (1+i) d_0 - d_1 - U^1 \right] \left( (1+i) d_1 - 1 \right) \]

We adopt the following parameters: \( U^0 = 100, i = 3\%, T = 1 \) year, \( \sigma = 30\% \), and let vary \( n_{\text{max}} \), the number of periods \( \Delta t \):

<table>
<thead>
<tr>
<th>( n_{\text{max}} )</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Price</td>
<td>0.077357</td>
<td>0.089813</td>
<td>0.092629</td>
<td>0.095299</td>
<td>0.095487</td>
<td>0.095497</td>
<td>0.095501</td>
<td>0.095502</td>
<td>0.095503</td>
<td>0.095504</td>
</tr>
</tbody>
</table>

\( T = 1 \quad \sigma = 15\% \)

<table>
<thead>
<tr>
<th>( n_{\text{max}} )</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Price</td>
<td>0.156573</td>
<td>0.176192</td>
<td>0.184964</td>
<td>0.190288</td>
<td>0.190712</td>
<td>0.190735</td>
<td>0.190743</td>
<td>0.190747</td>
<td>0.190749</td>
<td>0.190751</td>
</tr>
</tbody>
</table>

\( T = 1 \quad \sigma = 30\% \)

<table>
<thead>
<tr>
<th>( n_{\text{max}} )</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Price</td>
<td>0.391144</td>
<td>0.480611</td>
<td>0.493198</td>
<td>0.505915</td>
<td>0.506827</td>
<td>0.506873</td>
<td>0.506888</td>
<td>0.506896</td>
<td>0.506900</td>
<td>0.506903</td>
</tr>
</tbody>
</table>

The call price is therefore rather low but it significantly increases with the future maturity:

<table>
<thead>
<tr>
<th>( n_{\text{max}} )</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Price</td>
<td>0.811766</td>
<td>0.940916</td>
<td>0.976305</td>
<td>1.00738</td>
<td>1.00959</td>
<td>1.00970</td>
<td>1.00974</td>
<td>1.00976</td>
<td>1.00977</td>
<td>1.00978</td>
</tr>
</tbody>
</table>

\( T = 3 \quad \sigma = 15\% \)

<table>
<thead>
<tr>
<th>( n_{\text{max}} )</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
</table>

\( T = 10 \quad \sigma = 15\% \)

<table>
<thead>
<tr>
<th>( n_{\text{max}} )</th>
<th>2</th>
<th>3</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
</table>
This result prompts to focus on long-term contracts of the three month eurodollar type which span up to ten years.

Lastly it stands out that the option value approximately doubles as volatility switches from 15 to 30%.

**III.3.2 Valuation with Transaction Costs**

We take up the following parameters: \( U^0 = 100, \ i = 3\%, \ T = 1 \text{ year}, \ \sigma = 30\%, \) and the following transaction costs: interest rate spread = \( i_a - i_b = 2\text{bp}, \) underlying bid-ask spread = \( 2f = 0.10\%, \) future bid-ask spread = \( F_a - F_b = s = 1 \text{ tick} = 0.01. \) The so chosen fees are representative of the major liquid markets.

A negative price implies that the operator has no interest in initiating a dynamic C&C, he must be paid off to enter it.

<table>
<thead>
<tr>
<th>( \sigma = 15% )</th>
<th>( T = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_{\text{max}} )</td>
<td>2</td>
</tr>
<tr>
<td>Future locus at ( t_0 )</td>
<td></td>
</tr>
<tr>
<td>C&amp;C adjusted boundary</td>
<td>-0.01457</td>
</tr>
<tr>
<td>Middle of NSPAO range</td>
<td>-0.07784</td>
</tr>
<tr>
<td>RCC adjusted boundary</td>
<td>-0.14179</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma = 30% )</th>
<th>( T = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_{\text{max}} )</td>
<td>2</td>
</tr>
<tr>
<td>Future locus at ( t_0 )</td>
<td></td>
</tr>
<tr>
<td>C&amp;C adjusted boundary</td>
<td>0.05448</td>
</tr>
<tr>
<td>Middle of NSPAO range</td>
<td>-0.00439</td>
</tr>
<tr>
<td>RCC adjusted boundary</td>
<td>-0.06326</td>
</tr>
</tbody>
</table>

The transaction costs have considerably decreased the option price for the nearby maturities: it already was not worth a big deal in the absence of fees, with them now it is worth virtually nothing. Dynamic pure arbitrage is thus very little attractive for short maturities.

On the other hand for the long-dated contracts, like the three month libor eurodollar futures liquid up to ten years, the option still has value, as witness the following results.
If the option was integrated in the market prices, arbitrageurs would set the C&C threshold as the ask of the forward quote minus the call value, — minus for a C&C requires selling the future — . This would scale down accordingly the free fluctuation range of the future — which would then prevent also the opportunities of pure dynamic arbitrage —
and that in a broader sense we keep calling NPAO range.

A same argument is applied to the RCC threshold and allows to get a new representation of the No Pure Arbitrage Opportunity range:

Sketch 4 illustrates that ceteris paribus, the future dominates further more clearly the forward in terms of price than what Sketch 1 displays. Indeed the future is bought cheaper than the forward ask by at least the call value, and is sold above the forward bid by at least the put value.

But the present study unveils another, very astonishing outcome: when the future maturity is remote the call values compiled in the previous tables often amply exceed the NPAO range width which remains small as shown in the table below.

<table>
<thead>
<tr>
<th>T</th>
<th>1</th>
<th>3</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPAO Range Width</td>
<td>0.2260</td>
<td>0.2822</td>
<td>0.5297</td>
</tr>
<tr>
<td>Adjusted NPAO Range Width</td>
<td>0.2060</td>
<td>0.2185</td>
<td>0.2688</td>
</tr>
</tbody>
</table>

Acknowledging this implies that such long maturity future prices can never be efficient. Indeed if the long-dated future is efficient in the static pure arbitrage sense, it must lie between $F_a^{FWD}$ and $F_b^{FWD}$, but then it proves not efficient in the dynamic C&C sense since located above $F_a^{FWD} - C$, and it is as well not efficient in the dynamic RCC sense for located below $F_b^{FWD} + P$.

We entitle this behavior "dynamic boundaries crossing" and exhibit it in Sketch 5 hereafter.

Conversely if such a future is efficient in the dynamic pure arbitrage sense, then it offers a static C&C or RCC opportunity. Therefore, owing to its auction based working system and to its fungibility, the future has, for long maturities, an ever inefficient price.
IV Conclusion

In this paper we have avoided the four main critiques directed at the work of Brennan and Schwartz, which 1) supposes that the futures are not fungible whereas they are by definition, 2) needs two options in its setting whereas unwinding early brings in objectively only one, 3) gets without demonstration an apparently inappropriate stochastic differential equation, and 4) denies the endogeneous nature of this option by forcing the static pure arbitrage payoff to follow an ex cathedra process called brownian bridge.

Quite on the contrary, this paper complies with and exploits the future fungibility, notably in the price comparison with the forward, fully characterizes the underlying asset and the cash flows of the implicit unwinding option, proves all results, and derives in an endogeneous way the process followed by the arbitrage payoff given those followed by the underlying asset and the future basis, net of its carry.

In other respects, the transaction costs conveyed by each financial instrument, as well as the arbitrage engineering and management here expressed formally, reflect what is witnessed and performed in the markets, a requirement that Brennan and Schwartz wish it were fulfilled because they have been compelled first to suppose that operators trigger off their arbitrages in a frictionless world and second to add up a lump-sum cost, which provides a model very far from market reality.

Because of the lengthy setting up of the study framework and pricing trees, we restricted ourselves to value numerically the implicit early unwinding option in the sole cash-and-carry case. For nearby future maturities, the price in the absence of fees, arrived at with a risk-neutral valuation on a tree, falls dramatically when transaction costs are introduced.
On the other hand, for far remote maturities like those of the three month libor eurodollar contracts, the option price remains significant despite fees, which is an argument favoring the futures price dominance over forwards. Incidentally an attractive feature is that the implicit option price increases so much with the maturity that the futures constantly offer arbitrage opportunities, say it another way they become inefficient.

REFERENCES


In order to avoid that $\lambda$ lies outside of the $[0;1]$ interval as the future reaches the boundaries of the NSPAO range, two other processes have been explored. The former assumes that the net buying and selling bases can come closer and closer to their minimal level $s$ (in absolute value) following a geometric suite, but can never reach it. We expand here the net buying basis model, of simpler presentation for this basis is positive — the selling model deducing easily from it — :

\[
\begin{align*}
0_{a BN} &= (1-\delta)(0_{a BN} - s) \\
\pi \lambda \quad BN_a &= (1-\delta)(0_{a BN} - s) + \delta (0_{a BN} - s) \\
(1-\pi) \lambda \quad BN_a &= (1-\delta)(0_{a BN} - s) \\
(1-\pi)(1-\lambda) \quad BN_a &= ld - s + (1-\delta)(0_{a BN} - s) + \delta (ld - s)
\end{align*}
\]

$\delta$ = decrease rate of the net bases ($0 < \delta < 1$).

When the net buying basis decreases, the selling one increases.

Unfortunately this tree is little recombining ($4^n - 2(n+1)$ nodes) and, depending on the nodes, a number of net basis values embody a lot of terms inherited from the path taken thus far, which makes the calculations very cumbersome.

The latter alternative aims at eliminating the path dependence seen in the former. The net bases are expressed in relation with the NSPAO range width:
By furthering this model to any subsequent node it comes out that the net basis depends only on variables valued at this node and not before. Unfortunately it comes out likewise that the net basis values are not redundant enough to avoid a rapid growth of the nodes \((3 \times 4^{n-1}\) nodes).

We leave for future work the task of implementing these hefty size trees. Nevertheless endeavours to devote will not be well paid in return for, with the bid-ask spreads witnessed in major markets, the contribution of the net basis fluctuations to dynamic arbitrage is in general far much less than the underlying moves‘.